

# Confidence Intervals

From sampling uncertainty to a range of plausible values

Sergio I. García-Ríos

PA 311N: Data Analysis for Public Policy

LBJ School of Public Affairs

October 12, 2026

## Where we left off

Last time we learned that a sample statistic varies from sample to sample.

$\bar{x}$  changes, but its sampling distribution is predictable.

For a sample mean,

$$SE(\bar{x}) \approx \frac{s}{\sqrt{n}}.$$

Today we use that sampling uncertainty to move from a **single estimate** to a **range of plausible values** for the population parameter.

## A point estimate hides uncertainty

Suppose a random sample of 50 Ames homes gives

$$\bar{x} = 1520 \text{ sq. ft.}$$

Is the population mean exactly 1520 square feet?

No. Another random sample would probably give a different  $\bar{x}$ .

### The inferential question

How far from the sample estimate might the true population mean reasonably be?

**Estimate  $\pm$  uncertainty**

# The general confidence-interval architecture

A confidence interval has two pieces:

$$\text{point estimate} \pm \text{margin of error}$$

and

$$\text{margin of error} = \text{critical value} \times \text{standard error}$$

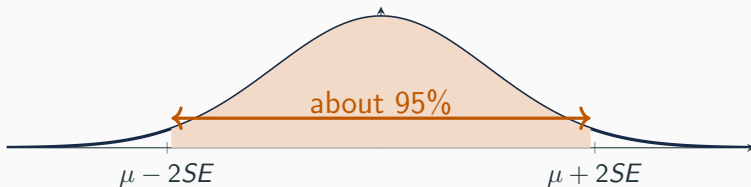
So the generic form is

$$\text{estimate} \pm (\text{critical value})(SE)$$

The estimate tells us where to center the interval. The margin of error tells us how far to extend it.

## Why about two standard errors?

If the sampling distribution is approximately normal,



About 95% of sample means fall within roughly 2 standard errors of  $\mu$ .

Turn that logic around: an interval extending roughly 2 SEs around  $\bar{x}$  will capture  $\mu$  about 95% of the time.

## What 95% confidence means

Imagine repeating the entire study many times:

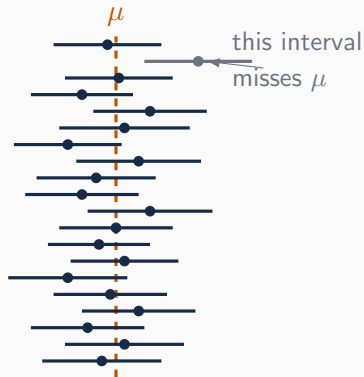
1. draw a random sample;
2. calculate a sample mean;
3. build a 95% confidence interval using the same procedure.

About 95% of those intervals would contain the true population mean.

### **Confidence is about the procedure**

The population parameter is fixed. The interval changes from sample to sample.

## Repeated intervals: most capture $\mu$ , some miss



different random samples  $\rightarrow$  different intervals

A 95% method is allowed to miss sometimes.

## A common misinterpretation

### Incorrect

“There is a 95% probability that  $\mu$  lies inside this particular interval.”

In the frequentist framework, after we calculate the interval:

- ▶  $\mu$  is fixed;
- ▶ the interval either contains  $\mu$  or it does not.

A useful interpretation is:

We are **95% confident** because this procedure captures the true parameter in about 95% of repeated random samples.

## Confidence interval for a population mean

If the population standard deviation  $\sigma$  were known,

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}.$$

But in real applications,  $\sigma$  is almost never known.

We estimate it with the sample standard deviation  $s$ :

$$SE(\bar{x}) \approx \frac{s}{\sqrt{n}}.$$

That extra estimation uncertainty changes the critical value we use.

## When $\sigma$ is unknown: use the $t$ distribution

For a confidence interval for a population mean,

$$\bar{x} \pm t_{df}^* \frac{s}{\sqrt{n}}$$

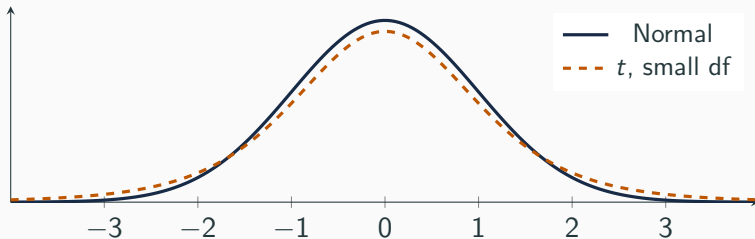
where

$$df = n - 1.$$

The  $t$  distribution is like the normal distribution, but with heavier tails.

For large samples,  $t$  and  $z$  become very similar.

## $t$ vs. normal



The heavier tails make  $t^*$  slightly larger than  $z^*$  when  $n$  is small.

## Critical values depend on confidence level

For a normal distribution:

| Confidence level | Central area | $z^*$ |
|------------------|--------------|-------|
| 90%              | 0.90         | 1.645 |
| 95%              | 0.95         | 1.960 |
| 99%              | 0.99         | 2.576 |

For a mean with unknown  $\sigma$ , use the corresponding  $t^*$  for  $df = n - 1$ .

More confidence requires a larger critical value.

Confidence intervals based on the CLT inherit the same conditions we discussed last week.

### 1. **Independence / random sampling**

- random sampling helps;
- if sampling without replacement,  $n$  should be less than about 10% of the population.

### 2. **Nearly normal sampling distribution**

- population roughly normal, or
- sample large enough that the CLT works well;
- more skewness/outliers require a larger  $n$ .

A formula cannot repair a badly designed sample.

**Let's build one from start to finish.**

## Ames example: one sample of 50 homes

Suppose an illustrative random sample gives

$$\bar{x} = 1520 \text{ sq. ft.}, \quad s = 500 \text{ sq. ft.}, \quad n = 50.$$

We want a 95% confidence interval for the population mean living area  $\mu$ .

Because  $\sigma$  is unknown:

$$df = 50 - 1 = 49.$$

The interval is

$$\bar{x} \pm t_{49}^* \frac{s}{\sqrt{50}}.$$

## Step 1: calculate the standard error

$$SE(\bar{x}) = \frac{s}{\sqrt{n}} = \frac{500}{\sqrt{50}} \approx \boxed{70.71 \text{ sq. ft.}}$$

Interpretation:

Across repeated random samples of 50 homes, sample means typically vary by about **71 square feet** around the population mean.

## Step 2: find the critical value

For a 95% confidence interval with  $df = 49$ ,

$$t_{49}^* \approx \boxed{2.01}.$$

In R:

```
qt(0.975, df = 49)
```

Why 0.975?

95% in the middle  $\Rightarrow$  2.5% in each tail.

### Step 3: calculate the margin of error

$$ME = t^* \times SE$$

$$ME = 2.01(70.71) \approx \boxed{142.1 \text{ sq. ft.}}$$

The margin of error is the distance from the point estimate to either endpoint.

$$1520 \pm 142.1$$

## Step 4: construct the interval

$$1520 \pm 142.1$$

$$(1377.9, 1662.1)$$

### Interpretation

Using this method, we are 95% confident that the population mean above-ground living area of Ames homes lies between about 1,378 and 1,662 square feet.

For this teaching dataset, the known population mean is 1499.69 square feet, which falls inside the interval.

## Your turn: identify every piece

### Your Turn

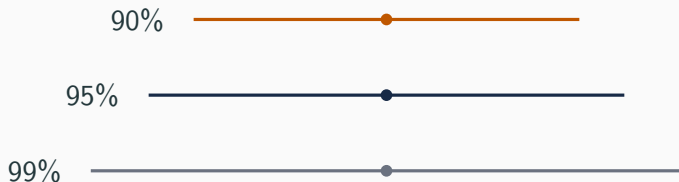
A sample of  $n = 64$  permit applications has mean processing time  $\bar{x} = 31.2$  days and standard deviation  $s = 8.0$  days.

For a 95% confidence interval:

1. What is the standard error?
2. What are the degrees of freedom?
3. What role does  $t^*$  play?
4. What would the margin of error represent in context?

## Confidence level and interval width

Holding the sample and standard error fixed:



higher confidence  $\Rightarrow$  larger critical value  $\Rightarrow$  wider interval

Confidence and precision trade off against each other when  $n$  is fixed.

## Sample size and interval width

Holding confidence level and variability fixed,

$$ME = t^* \frac{s}{\sqrt{n}}.$$

As  $n$  increases,

$$\frac{s}{\sqrt{n}} \downarrow \quad \Rightarrow \quad ME \downarrow .$$

Larger samples produce **narrower confidence intervals** because the point estimate is more precise.

But the relationship follows a square root, so gains become progressively more expensive.

# The square-root rule

Suppose you want to cut the margin of error in half.

Because

$$ME \propto \frac{1}{\sqrt{n}},$$

we need

$$\frac{1}{\sqrt{n_{new}}} = \frac{1}{2} \frac{1}{\sqrt{n_{old}}} \implies \boxed{n_{new} = 4n_{old}}.$$

## Your Turn

If a poll has a margin of error of about  $\pm 4$  points, roughly how much larger must the sample be to reach  $\pm 2$  points, all else equal?

## Planning sample size before collecting data

Using a target margin of error  $ME$  and an anticipated standard deviation  $\sigma$ ,

$$ME \approx z^* \frac{\sigma}{\sqrt{n}}.$$

Solve for  $n$ :

$$n \approx \left( \frac{z^* \sigma}{ME} \right)^2$$

This is a planning approximation. In practice,  $\sigma$  may come from prior data or a pilot study.

Always round the required sample size up.

## Policy planning example: how many homes?

Suppose the city wants a 95% confidence interval for mean home size with margin of error no larger than 50 square feet.

Assume prior data suggest  $\sigma \approx 500$  square feet.

$$n \approx \left( \frac{1.96(500)}{50} \right)^2 = 384.16.$$

So the city should plan for at least

385 homes.

This turns a statistical precision requirement into a concrete data-collection decision.

## Your turn: loosen the precision requirement

### Your Turn

Same setting: 95% confidence and anticipated  $\sigma = 500$  square feet.

What sample size would be needed if the city is willing to accept a margin of error of  $\pm 100$  square feet?

$$n \approx \left( \frac{1.96(500)}{100} \right)^2 = 96.04.$$

Round up:

$$\boxed{n = 97}.$$

## Three levers determine interval width

| Change             | Effect on width | Why                      |
|--------------------|-----------------|--------------------------|
| Higher confidence  | Wider           | critical value increases |
| More variability   | Wider           | standard error increases |
| Larger sample size | Narrower        | standard error decreases |

Width is not mysterious. It comes from the formula.

## Common misconceptions

1. “95% of individual homes are inside the confidence interval.”

**No.** The interval estimates the *population mean*, not the spread of individual homes.

2. “A 99% interval is more precise than a 95% interval.”

**No.** With the same data, 99% confidence produces a wider, less precise interval.

3. “A narrow interval means the study is unbiased.”

**No.** A large biased sample can be very precise about the wrong value.

## Confidence intervals quantify sampling uncertainty, not everything

A confidence interval reflects uncertainty generated by the sampling process under the model.

It does **not** automatically account for:

- ▶ coverage error;
- ▶ nonresponse bias;
- ▶ measurement error;
- ▶ a systematically unrepresentative sample;
- ▶ bad causal assumptions.

More decimal places cannot rescue bad data.

## Using an interval for a policy decision

Suppose a city service has a target mean processing time of 30 days.

A study estimates

$$\bar{x} = 29.6 \text{ days}, \quad 95\% \text{ CI} = (27.4, 31.8).$$

### Your Turn

Can we confidently conclude that the true mean processing time is below 30 days?

No. The interval includes values both below and above the 30-day target.

This logic is the bridge to **hypothesis testing**.

# The workflow for a confidence interval

1. **Name the parameter.** What population quantity are we estimating?
2. **Check the conditions.** Is the sampling/model setup defensible?
3. **Calculate the point estimate.**
4. **Calculate the standard error.**
5. **Choose the critical value.**
6. **Construct the interval.**
7. **Interpret it in context.**

Estimate  $\rightarrow$  Uncertainty  $\rightarrow$  Interval  $\rightarrow$  Interpretation

## What to remember

1. A point estimate without uncertainty is incomplete.
2. Confidence intervals are built from

$$\text{estimate} \pm (\text{critical value})(\text{SE}).$$

3. For a mean with unknown  $\sigma$ , use  $t^*$  and  $s/\sqrt{n}$ .
4. Higher confidence widens the interval; larger  $n$  narrows it.
5. 95% confidence describes long-run performance of the *procedure*.

See the estimate. Quantify the uncertainty. Interpret the range.